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(Residential Autonomous College affiliated to University of Calcutta)

B.A./B.Sc. FIFTH SEMESTER EXAMINATION, DECEMBER 2017
THIRD YEAR [BATCH 2015-18]
MATHEMATICS [Honours]
Paper : V

Date : 19/12/2017
Time : 11 am – 3 pm
Full Marks : 100

[Use a separate Answer Book for each Group]

<u>Group – A</u>

Answer <u>any five</u> questions from <u>Question Nos. 1 to 8</u> : [5×10]

1. a) Let G be a group of order 8 and x be an element of G of order 4. Prove that $x^2 \in Z(G)$.
   b) If H be a subgroup of a cyclic group G, then prove that the quotient group G/H is cyclic. Is the converse true? Justify your answer.
   c) Let α and β be two group homomorphisms from G to G' and let $H = \{g \in G \mid \alpha(g) = \beta(g)\}$. Prove or disprove H is a subgroup of G. 3+5+2

2. a) Let G be an abelian group of order 8. Prove that $\phi: G \to G$ defined by $\phi(x) = x^3 \forall x \in G$ is an isomorphism.
   b) Let G be a group and A, B are subgroups of G. If (i) $G = AB$, (ii) $ab = ba$ for all $a \in A, b \in B$ and (iii) $A \cap B = \{e\}$ prove that G is an internal direct product of A and B. Hence show that Klein's 4-group is isomorphic to the internal direct product of a cyclic group of order 2 with itself.
   c) Write class equation for a finite group G. 3+5+2

3. a) If G is a finite commutative group of order n such that n is divisible by a prime p, then prove that G contains an element of order p.
   b) Prove that no group of order 56 is simple. 5+5

4. a) State and prove Sylow's 3rd theorem.
   b) If $o(G) = p^n$ where p is prime, $n > 0$; prove that $Z(G)$ is nontrivial.
   c) Use (b) above to show that a group of order p^2 where p is prime is abelian. 5+3+2

5. a) Find all ideals of the ring $(□, +, \cdot)$.
   b) Let $T_2(□) = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \mid a, b, c \in □ \right\}$
   be the ring of all upper triangular matrices over □. Prove that $I = \left\{ \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} \mid a \in □ \right\}$ is an ideal of $T_2(□)$. Find the quotient ring $T_2(□)/I$.
   c) Find all automorphisms of the field □. 2+3+5

6. a) Define a Euclidean Domain. Give example of it. Prove that every field is a Euclidean domain.
   b) In an integral domain, prove that every prime element is irreducible.
   c) In $\mathbb{Z}_6$, prove that [3] is prime, but not irreducible. 4+3+3

7. a) Prove that in a UFD, every irreducible element is prime.

b) Show that in the integral domain $□\left[i\sqrt{5}\right]$, 3 is irreducible but not prime. 5+5

8. a) Let R be a commutative ring with 1. Prove that every proper ideal of R is contained in a maximal ideal of R.

b) Let I be a prime ideal in R and $a,b \in R-I$ then prove that there exists $c \in R$ such that $acb \in R-I$.

c) In $C([0,1])$, let $M_{\frac{1}{2}} = \left\{ f \in C([0,1]) : f\left(\frac{1}{2}\right) = 0 \right\}$. Show that $M_{\frac{1}{2}}$ is a maximal ideal of $C([0,1])$. 5+2+3

Group – B

Answer <u>any six</u> questions from <u>Question Nos. 9 to 17</u> **:** [6×5]

9. a) Calculate the partial derivatives f_{xy} and f_{yx} at the point (1, 2) for the function [2]

$$f(x,y) = \begin{cases} (x-1)(y-2)^2 & , \text{ if } y > 2 \\ -(x-1)(y-2)^2 & , \text{ if } y \le 2 \end{cases}$$

b) Find the double and repeated limits of the function $f(x,y) = \begin{cases} (x+y)\sin\frac{1}{x} & , \text{ if } x \ne 0 \\ 0 & , \text{ if } x = 0 \end{cases}$

as x and y tend to 0. [3]

10. Let $f : U \to □$, where $U \subseteq □^2$ is an open set. Let $(x_0, y_0) \in U$ and f(x,y) satisfies

i) $\frac{\partial f}{\partial x}$ exists in some neighbourhood of (x_0,y_0)

ii) $\frac{\partial^2 f}{\partial x \partial y}$ is continuous at (x_0,y_0).

Show that $\frac{\partial^2 f}{\partial y \partial x}$ exists at (x_0,y_0) and $\frac{\partial^2 f}{\partial y \partial x}(x_0, y_0) = \frac{\partial^2 f}{\partial x \partial y}(x_0, y_0)$. [5]

11. If $u^3 = xyz$, $\frac{1}{v} = \frac{1}{x} + \frac{1}{y} + \frac{1}{z}$, $w^2 = x^2 + y^2 + z^2$, prove that

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \frac{-v(y-z)(z-x)(x-y)(x+y+z)}{3u^2 w(yz+zx+xy)}.$$ [5]

12. Show that the function f(x,y) defined by

$$f(x,y) = \frac{xy}{\sqrt{x^2+y^2}}, \quad x^2 + y^2 \ne 0$$

$$= 0, \qquad x = y = 0$$

is continuous, possesses partial derivatives of first order but is not differentiable at origin. [1+2+2]

13. a) Apply Lagrange's M.V.T. for a function f(x,y) of two variables given by $f(x,y) = \sin \pi x + \cos \pi y$ to express $f\left(\frac{1}{2}, 0\right) - f\left(0, -\frac{1}{2}\right)$ in terms of first order partial derivatives of f and show that $\exists$ a real no. θ where $0 < \theta < 1$ s.t. $\frac{4}{\pi} = \cos\frac{\pi\theta}{2} + \sin\frac{\pi}{2}(1-\theta)$. [2]

b) Find the Taylor expansion of cos (x+y) upto second degree terms (excluding remainder) about the point $\left(1, \frac{\pi}{2}\right)$. [3]

14. a) Find the maximum value of the function $f(x,y) = \sin x \sin y \sin(x+y)$ defined in the triangular region $0 \le x \le \pi$, $0 \le y \le \pi$, $0 \le x + y \le \pi$. [3]

b) Check the independence of the functions $f_1(x,y,z) = -x + y + z$, $f_2(x,y,z) = x - y + z$ and $f_3(x,y,z) = x^2 + y^2 + z^2 - 2xy$. If they are dependent, find the relation between them. [2]

15. Let $f : \mathbb{R}^3 \to \mathbb{R}^2$ be a function of the form $f(x,y,z) = (f_1(x,y,z), f_2(x,y,z))$. Show that f is a differentiable function iff f_1, f_2 are differentiable. [5]

16. Let $f(x,y) = y^2 - yx^2 - 2x^5$. Check whether it is possible to solve f(x,y) = 0 uniquely in some neighbourhood of (1, –1). If yes, then find the solution and $\frac{dy}{dx}$ at (1,–1). [5]

17. a) State the sufficient condition for the continuity of a function $f : \mathbb{R}^2 \to \mathbb{R}$. [1]

b) If $f(x,y) = \begin{cases} \frac{2xy}{x^2+y^2} & , \text{ if } x^2 + y^2 \ne 0 \\ 0 & , \text{ otherwise} \end{cases}$

show that both the first order partial derivatives exists at (0,0) but f(x,y) is discontinuous there. Does this violate the sufficient condition for the continuity as stated in problem (17a)? [4]

Answer <u>any four</u> questions from <u>Question Nos. 18 to 23</u> : [4×5]

18. Define a function of bounded variation. Show that the function $f : [0,1] \to \mathbb{R}$ defined by

$$f(x) = x \sin\frac{\pi}{x}, \quad x \in (0,1]$$
$$= 0, \quad x = 0$$

is a bounded function but it is not a function of bounded variation. [2+1+2]

19. Show that the plane curve γ defined by $\gamma(x) = (f(x), g(x))$, $x \in [0,1]$

where $f(x) = x^2 \quad 0 \le x \le 1$

& $g(x) = x^2 \sin\frac{1}{x}, \quad 0 < x \le 1$

$= 0, \quad x = 0$

is rectifiable on [0,1] [5]

20. State Bonnet's form of 2^{nd} M.V.T. of integral calculus. Use it to show that $\left|\int_a^b \frac{\sin x}{x} dx\right| \le \frac{2}{a}$ if $b > a > 0$. [2+3]

21. a) Evaluate : $\lim_{x \to 0} \frac{x\int_0^x e^{t^2} dt}{1 - e^{x^2}}$. [2]

b) A function f is defined over the closed interval [1, 3] as follows

$$f(x) = 1, \ 1 \le x < 2$$
$$= 2, \ 2 \le x \le 3$$

Verify whether $\int_a^b f(x)dx = (b-a)f(\xi)$ holds here for some $\xi \in [a, b]$. [3]

22. Show that $\frac{\pi^3}{96} < \int_{-\pi/2}^{\pi/2} \frac{x^2}{5 + 3\sin x} dx < \frac{\pi^3}{24}$. [5]

23. A function $f : [a, b] \to \mathbb{R}$ be integrable on [a,b]. The function F is defined by $F(x) = \int_a^x f(t)dt$, $x \in [a, b]$. Prove that if f is continuous at $c \in [a, b]$ then F is differentiable at c and $F'(c) = f(c)$. [5]

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